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Numerical Solution of Optimal Control Problems by Direct Collocation

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Abstract. By an appropriate discretization of control and state variables, a constrained optimal control problem is transformed into a finite dimensional nonlinear program which can be solved by standard SQP-methods [10]. Convergence properties of the discretization are derived. From a solution of this method known as direct collocation, these properties are used to obtain reliable estimates of adjoint variables. In the presence of active state constraints, these estimates can be significantly improved by including the switching structure of the state constraint into the optimization procedure. Two numerical examples are presented.

1 Statement of problems

Systems governed by ordinary differential equations arise in many applications as, e. g., in astronautics, aeronautics, robotics, and economics. The task of optimizing these systems leads to the optimal control problems investigated in this paper.

The aim is to find a control vector $u(t)$ and the final time t_f that minimize the functional

$$J[u, t_f] = \Phi(x(t_f), t_f) \tag{1}$$

subject to a system of n nonlinear differential equations

$$\dot{x}_i(t) = f_i(x(t), u(t), t), \quad i = 1, \ldots, n, \quad 0 \le t \le t_f, \tag{2}$$

boundary conditions

$$r_i(x(0), x(t_f), t_f) = 0, \quad i = 1, \ldots, k \le 2n, \tag{3}$$

and m inequality constraints

$$g_i(x(t), u(t), t) \ge 0, \quad i = 1, \ldots, m, \quad 0 \le t \le t_f. \tag{4}$$

Here, the l vector of control variables is denoted by $u(t) = (u_1(t), \ldots, u_l(t))^T$ and the n vector of state variables is denoted by $x(t) = (x_1(t), \ldots, x_n(t))^T$. The functions $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}$, $f : \mathbb{R}^{n+l+1} \to \mathbb{R}^n$, $r : \mathbb{R}^{2n+1} \to \mathbb{R}^k$, and $g : \mathbb{R}^{n+l+1} \to \mathbb{R}^m$ are assumed to be continuously differentiable. The controls $u_i : [0, t_f] \to \mathbb{R}$, $i = 1, \ldots, l$, are assumed to be bounded and measureable and t_f may be fixed or free.

2 Discretization

This section briefly recalls the discretization scheme as described in more detail in [18]. Some of the basic ideas of this discretization scheme have been formerly outlined by Kraft [14] and Hargraves and Paris [11].
A discretization of the time interval

$$0 = t_1 < t_2 < \ldots < t_N = t_f \tag{5}$$

is chosen. The parameters Y of the nonlinear program are the values of control and state variables at the grid points $t_j,\ j = 1, \ldots, N$, and the final time t_f

$$Y = (u(t_1), \ldots, u(t_N), x(t_1), \ldots, x(t_N), t_N) \in \mathbb{R}^{N(l+n)+1}. \tag{6}$$

The controls are chosen as piecewise linear interpolating functions between $u(t_j)$ and $u(t_{j+1})$ for $t_j \leq t < t_{j+1}$

$$u_{\text{app}}(t) = u(t_j) + \frac{t - t_j}{t_{j+1} - t_j}(u(t_{j+1}) - u(t_j)). \tag{7}$$

The states are chosen as continuously differentiable functions and piecewise defined as cubic polynomials between $x(t_j)$ and $x(t_{j+1})$ with $\dot{x}_{\text{app}}(s) := f(x(s), u(s), s)$ at $s = t_j,\ t_{j+1}$,

$$\begin{aligned}
x_{\text{app}}(t) &= \sum_{k=0}^{3} c_k^j \Big(\frac{t - t_j}{h_j}\Big)^k, \quad t_j \leq t < t_{j+1}, \quad j = 1, \ldots, N-1, && (8)\\
c_0^j &= x(t_j), && (9)\\
c_1^j &= h_j f_j, && (10)\\
c_2^j &= -3x(t_j) - 2h_j f_j + 3x(t_{j+1}) - h_j f_{j+1}, && (11)\\
c_3^j &= 2x(t_j) + h_j f_j - 2x(t_{j+1}) + h_j f_{j+1}, && (12)\\
\text{where } f_j &:= f(x(t_j), u(t_j), t_j), \quad h_j := t_{j+1} - t_j.
\end{aligned}$$

The approximating functions of the states have to satisfy the differential equations (2) at the grid points $t_j,\ j = 1, \ldots, N$, and at the centers $t_{c,j} := t_{j+1/2} := (t_j + t_{j+1})/2$, $j = 1, \ldots, N-1$, of the discretization intervals. This scheme is also known as cubic collocation at Lobatto points. The chosen approximation (8) – (12) of $x(t)$ already fulfills these constraints at t_j. Therefore, the only remaining constraints in the nonlinear programming problem are

- the collocation constraints at $t_{c,j}$

$$f(x_{\text{app}}(t_{c,j}), u_{\text{app}}(t_{c,j}), t_{c,j}) - \dot{x}_{\text{app}}(t_{c,j}) = 0,\ j = 1, \ldots, N-1, \tag{13}$$

- the inequality constraints at the grid points t_j

$$g(x_{\text{app}}(t_j), u_{\text{app}}(t_j), t_j) \geq 0,\ j = 1, \ldots, N, \tag{14}$$

- and the initial and end point constraints at t_1 and t_N

$$r(x_{\text{app}}(t_1), x_{\text{app}}(t_N), t_N) = 0. \tag{15}$$

In the following, the index "app" for approximation will be suppressed.
By this scheme the number of four free parameters for each cubic polynomial is reduced to two and the number of three collocation constraints per subinterval is reduced to one. Compared with other collocation schemes we have a reduced number of constraints to be fulfilled and a reduced number of free parameters to be determined by the numerical procedure. This results in a better performance of an implementation of this method in terms of convergence, reliability, and efficiency compared with other schemes.

3 Convergence properties of the discretization

In the sequel, we assume that, for example, the controls $u_i,\ i = 1, \dots, l$, appear nonlinearly in f, the optimal control is continuous and the final time t_f is fixed. Furthermore, we assume that the number of inequality constraints m is 1 and that the constraint $g = g_1$ is active within an interval $[t_{\text{entry}}, t_{\text{exit}}]$ along the optimal trajectory, where $0 < t_{\text{entry}} < t_{\text{exit}} < t_f$.

3.1 Necessary first order optimality conditions of the continuous problem

There exist an n-vector function of adjoint or costate variables $\lambda(t) = (\lambda_1(t), \dots, \lambda_n(t))^T$ and a multiplier function $\eta(t)$. With the Hamiltonian

$$H(x, u, t, \lambda, \eta) = \sum_{k=1}^{n} \lambda_k f_k(x, u, t) \ + \ \eta(t) g(x, u, t), \tag{16}$$

the necessary first order conditions of optimality result in a multi-point boundary value problem

$$\begin{aligned}
\dot{x}_i(t) &= \frac{\partial H}{\partial \lambda_i} = f_i(x, u, t), && (17)\\
\dot{\lambda}_i(t) &= -\frac{\partial H}{\partial x_i} = -\sum_{k=1}^{n} \lambda_k(t) \frac{\partial f_k(x, u, t)}{\partial x_i} - \eta(t) \frac{\partial g(x, u, t)}{\partial x_i}, \ i = 1, \dots, n, && (18)\\
0 &= \frac{\partial H}{\partial u_j} = \sum_{k=1}^{n} \lambda_k(t) \frac{\partial f_k(x, u, t)}{\partial u_j} + \eta(t) \frac{\partial g(x, u, t)}{\partial u_j}, \ j = 1, \dots, l, && (19)
\end{aligned}$$

$$g(x, u, t) > 0 \text{ and } \eta(t) = 0, \quad \text{or} \quad g(x, u, t) = 0 \text{ and } \eta(t) \le 0. \tag{20}$$

The original boundary constraints (3) and additional constraints on $\lambda(t)$ at 0, t_{entry}, t_{exit}, and t_f also have to be fulfilled. In general, at junction points t_{entry}, t_{exit}, the adjoint variables may have discontinuities. For more details cf. Bryson, Ho [4] and Hestenes [12] and also Jacobson, Lele, Speyer [13], Maurer [15], and the results of

Maurer cited in Bulirsch, Montrone, Pesch [5] for the necessary conditions of optimality in the constrained case.
In the sequel, we shall see that the necessary first order optimality conditions of the continuous problem are reflected in the necessary first order optimality conditions of the discretized problem.

3.2 Necessary first order optimality conditions of the discretized problem

For the sake of simplicity, we now assume that $n = 1$ and $l = 1$. In this section, we will use the notations

$$u_i \;:=\; u(t_i), \quad x_i \;:=\; x(t_i), \quad i = 1, \ldots, N, \tag{21}$$

and

$$f_i \;:=\; f(x_i, u_i, t_i), \quad f_{i+1/2} \;:=\; f(x(t_{i+1/2}), u(t_{i+1/2}), t_{i+1/2}). \tag{22}$$

The Lagrangian of the nonlinear program of the discretized problem from Sec. 2 can then be written as

$$\begin{aligned} L(Y, \mu, \sigma, \nu) \;=\;& \Phi(x_N, t_N) \\ & - \sum_{j=1}^{N-1} \mu_j \left(f(x(t_{c,j}), u(t_{c,j}), t_{c,j}) - \dot{x}(t_{c,j}) \right) \\ & - \sum_{j=1}^{N} \sigma_j g(x_j, u_j, t_j) - \sum_{j=1}^{k} \nu_j r_j(x(t_1), x(t_N), t_N) \end{aligned} \tag{23}$$

with $\mu = (\mu_1, \ldots, \mu_{N-1})^T \in \mathbb{R}^{N-1}$, $\sigma = (\sigma_1, \ldots, \sigma_N)^T \in \mathbb{R}^N$ and $\nu = (\nu_1, \ldots, \nu_k)^T \in \mathbb{R}^k$. A solution of the nonlinear program fulfills the necessary first order optimality conditions of Karush, Kuhn, and Tucker, cf., e. g., [9]. Among others, these are

$$\frac{\partial L}{\partial u_i} = 0, \quad \frac{\partial L}{\partial x_i} = 0, \quad \frac{\partial L}{\partial \mu_j} = 0, \quad i = 1, \ldots, N, \; j = 1, \ldots, N-1. \tag{24}$$

$$g(x_i, u_i, t_i) > 0 \text{ and } \sigma_i = 0, \quad \text{or} \quad g(x_i, u_i, t_i) = 0 \text{ and } \sigma_i \leq 0. \tag{25}$$

As the "fineness" of the grid, we define

$$h \;:=\; \max\{h_j = t_{j+1} - t_j : \; j = 1, \ldots, N-1\}. \tag{26}$$

In detail, we find for $i = 2, \ldots, N$,

$$\begin{aligned} 0 \;=\; \frac{\partial L}{\partial u_i} \;=\;& -\mu_{i-1} \left(\frac{\partial f(x(t_{i-1/2}), u(t_{i-1/2}), t_{i-1/2})}{\partial u_i} - \frac{\partial \dot{x}(t_{i-1/2})}{\partial u_i} \right) \\ & -\mu_i \left(\frac{\partial f(x(t_{i+1/2}), u(t_{i+1/2}), t_{i+1/2})}{\partial u_i} - \frac{\partial \dot{x}(t_{i+1/2})}{\partial u_i} \right) \\ & -\sigma_i \frac{\partial g(x(t_i), u(t_i), t_i)}{\partial u_i} \end{aligned} \tag{27}$$

Using the basic relations (23) – (31) of [18] and the notation from (21), (22), we obtain after some calculations and by using the chain rule of differentiation

$$\begin{aligned}\frac{\partial L}{\partial u_i} &= -\frac{1}{2}\left(\frac{\partial f_{i-1/2}}{\partial u}\mu_{i-1} + \frac{\partial f_{i+1/2}}{\partial u}\mu_i\right) - \frac{1}{4}\frac{\partial f_i}{\partial u}(\mu_i + \mu_{i-1}) \\ &\quad + \frac{1}{8}\frac{\partial f_i}{\partial u}\left(h_{i-1}\mu_{i-1}\frac{\partial f_{i-1/2}}{\partial x} - h_i\mu_i\frac{\partial f_{i+1/2}}{\partial x}\right) - \sigma_i\frac{\partial g(x(t_i), u(t_i), t_i)}{\partial u}. \end{aligned} \tag{28}$$

Letting $h \to 0$ and keeping $t = t_i$ fixed, we have

$$\frac{\partial L}{\partial u_i} = -\frac{1}{2}\left(\frac{\partial f_i}{\partial u}\mu_i + \frac{\partial f_i}{\partial u}\mu_i\right) - \frac{1}{4}\frac{\partial f_i}{\partial u}(\mu_i + \mu_i) - \sigma_i\frac{\partial g(x(t_i), u(t_i), t_i)}{\partial u} \tag{29}$$

and finally

$$\frac{3}{2}\mu_i\frac{\partial f(x(t_i), u(t_i), t_i)}{\partial u} + \sigma_i\frac{\partial g(x(t_i), u(t_i), t_i)}{\partial u} = 0. \tag{30}$$

This equation is equivalent to the condition (19).
On the other hand, for $i = 2, \ldots, N-1$,

$$\begin{aligned} 0 = \frac{\partial L}{\partial x_i} &= -\mu_{i-1}\left(\frac{\partial f(x(t_{i-1/2}), u(t_{i-1/2}), t_{i-1/2})}{\partial x_i} - \frac{\partial \dot{x}(t_{i-1/2})}{\partial x_i}\right) \\ &\quad -\mu_i\left(\frac{\partial f(x(t_{i+1/2}), u(t_{i+1/2}), t_{i+1/2})}{\partial x_i} - \frac{\partial \dot{x}(t_{i+1/2})}{\partial x_i}\right) \\ &\quad -\sigma_i\frac{\partial g(x(t_i), u(t_i), t_i)}{\partial x_i}. \end{aligned} \tag{31}$$

Using again the basic relations (23) – (31) of [18] and the notation from (21), (22), we obtain after some calculations and by using the chain rule of differentiation

$$\begin{aligned}\frac{\partial L}{\partial x_i} &= -\frac{3}{2}\left(\frac{\mu_i}{h_i} - \frac{\mu_{i-1}}{h_{i-1}}\right) \\ &\quad -\frac{1}{4}\frac{\partial f_i}{\partial x}(\mu_{i-1} + \mu_i) - \frac{1}{2}\left(\mu_{i-1}\frac{\partial f_{i-1/2}}{\partial x} + \mu_i\frac{\partial f_{i+1/2}}{\partial x}\right) \\ &\quad + \frac{1}{8}\frac{\partial f_i}{\partial x}(h_{i-1}\mu_{i-1} - h_i\mu_i) - \sigma_i\frac{\partial g(x(t_i), u(t_i), t_i)}{\partial x}. \end{aligned} \tag{32}$$

For convenience, we now suppose an equidistant grid, i.e.

$$h = h_i = t_{i+1} - t_i = \frac{t_f - t_0}{N-1}, \quad i = 1, \ldots, N-1. \tag{33}$$

Now letting $h \to 0$ and keeping $t = t_i$ fixed, we have (cf. [18])

$$\frac{3}{2}\dot{\mu}_i + \frac{3}{2}\mu_i\frac{\partial f(x(t_i), u(t_i), t_i)}{\partial x} + \sigma_i\frac{\partial g(x(t_i), u(t_i), t_i)}{\partial x} = 0. \tag{34}$$

This equation is equivalent to the adjoint differential equation (18).
Similar results hold for a non-equidistant grid under additional conditions and for $n > 1$. They can also be extended to more general problems.

4 Estimates of adjoint variables

It has been shown in the previous section that the necessary conditions of optimality of the discretized problem reflect the necessary conditions of the original continuous problem. More precisely, it has been shown that Eq. (32) and Eq. (28), resp., are discretized versions of the adjoint differential equation (18) and the condition (19), respectively.

Therefore, we obtain an estimate of $\lambda(t)$ from the multipliers of the discretized problem by

$$\lambda(t_{i+1/2}) \;=\; -\frac{3}{2}\rho_i\,\mu_i, \quad i = 1, \ldots, N-1, \tag{35}$$

where ρ_i is a scaling factor depending on the discretization. In addition, an estimate of $\eta(t_i)$ can be obtained from σ_i.

Another approach for estimating adjoint variables in combination with a direct collocation method has been used by Enright and Conway [8]. They used the multipliers ν_j from Eq. (23) of the boundary conditions in the discretized problem in order to estimate $\lambda(t_f)$. This estimate is then used as an initial value for the backward integration of the adjoint differential equations (18). It is a well-known matter of fact that this backward integration is crucial for highly nonlinear problems. Also, state constraints were not considered.

A further approach for estimating adjoint variables is based on an interpretation of the adjoint variables as sensitivities connected to the gradient of the cost function

$$\lambda(t) \;=\; \frac{\partial \Phi}{\partial x}(t) \quad \text{at} \quad u \;=\; u_{\text{optimal}} \tag{36}$$

where x satisfies the differential equations (2). This relation can be found, e. g., in Breakwell [2] or in Bryson, Ho [4]. In a discretized version as, e. g.,

$$\lambda(t) \;=\; \frac{\tilde{\Phi}(x(t)+\delta) - \tilde{\Phi}(x(t)-\delta)}{2\,\delta} \quad \text{at} \quad u \;=\; \tilde{u}_{\text{optimal}}, \tag{37}$$

it can be used in combination with a direct shooting method as, e. g., [1], with a suitable steplength δ for the difference quotient. Here, the superscript $\tilde{}$ denotes that the variable or value has been obtained numerically, e. g., by a direct shooting method. For more details, cf., e. g., Eq. (29) in [1].

The guess of adjoint variables by direct methods is usually affected by several sources of inaccuracies and troubles. First, the suboptimal control $\tilde{u}_{\text{optimal}}$ calculated by a direct method is often inaccurate and can differ significantly from the optimal control. Second, the accuracy of the calculated objective $\tilde{\Phi}$ is often not better than one percent. In addition, the case of nearly active or inactive state variable inequality constraints has not yet been included in a reliable manner in previous attempts.

In contrast to the former approaches, the quality of the estimated adjoints does neither depend crucially on a highly accurate computation of the cost function or the calculated suboptimal control nor the appearance of active state constraints following our approach. As it is shown from the examples and the reported numerical results in this paper and in [6], [17], and [18], the new way of estimating adjoint variables herein proposed is very reliable and accurate even for complicated and highly nonlinear problems and problems including state constraints. Furthermore, convergence properties of the discretization scheme have been derived.

5 Examples and numerical results

The results reported in this section have been obtained by using the implementation DIRCOL (cf. [17]) of the direct collocation method mentioned in the previous sections. The use of grid refinement techniques yields a sequence of related nonlinear programs with increasing dimensions. In each macro iteration step, one nonlinear program has to be solved by the Sequential Quadratic Programming method NPSOL due to Gill, Murray, Saunders, and Wright [10]. The reported estimates of adjoint variables are direct outputs of DIRCOL.

5.1 Optimal ascent of the lower stage of a Sänger-type vehicle

This problem describes the lifting of an airbreathing lower stage of a two-stage-to-orbit Sänger-type launch vehicle. We focus on the Ramjet-powered second part of the trajectory. The four state variables are the velocity v, the flight path angle γ, the altitude h, and the mass m. The three control variables are the lift coefficient c_L, the thrust angle ϵ and the throttle setting δ, $\delta \in [0,1]$. The equations of motion are

$$\begin{aligned}
\dot{v} &= \frac{T(v,h;\delta)}{m}\cos\epsilon - \frac{D(v,h;c_L)}{m} - g(h)\sin\gamma, && (38)\\
\dot{\gamma} &= \frac{1}{v}\left(\frac{T(v,h;\delta)}{m}\sin\epsilon + \frac{L(v,h;c_L)}{m} - \left(g(h) - \frac{v^2}{r_0+h}\right)\cos\gamma\right), && (39)\\
\dot{h} &= v\sin\gamma, && (40)\\
\dot{m} &= b(v,h)\,\delta, \quad b(v,h) = \text{ maximum mass flow.} && (41)
\end{aligned}$$

The considered time interval is $[0, t_f]$ and t_f is free. The following formulae are used for the thrust, the lift and the drag forces

$$\begin{aligned}
T(v,h;\delta) &= T_m(v,h)\,\delta, \quad T_m(v,h) = \text{maximum thrust},\\
L(v,h;c_L) &= q(v,h)\,S\,c_L,\\
D(v,h;c_L) &= q(v,h)\,S\,c_D(m_a(v,h),c_L),\\
\text{where} \quad q(v,h) &= \frac{v^2}{2}\rho_0\exp(-\beta h), \quad g(h) \;=\; g_0\left(\frac{r_0}{r_0+h}\right)^2.
\end{aligned}$$

The lift and drag model has a quadratic polar

$$\begin{aligned} c_D(m_a(v,h),c_L) &= c_{D0}(m_a(v,h)) + k(m_a(v,h))\, c_L^2, \quad a(h) = \text{speed of sound}, \\ m_a(v,h) &= \frac{v}{a(h)}, \quad k(m_a) = \text{a characteristic function of the vehicle.} \end{aligned}$$

The quantities S, g_0, r_0, and ρ_0 are constants. For more details of the problem and for a three dimensional formulation cf. Chudej [7]. The boundary conditions are

$$\begin{array}{rclrcl} h(0) &=& 20 \text{ km}, & h(t_f) &=& 30 \text{ km}, \\ v(0) &=& 925 \text{ m/s}, & v(t_f) &=& 1700 \text{ m/s}, \\ \gamma(0) &=& 0.05, & \gamma(t_f) &=& 0.04, \\ m(0) &=& 332400 \text{ kg}. & & & \end{array} \tag{42}$$

The objective is to maximize the final mass, i. e.,

$$J[c_L, \epsilon, \delta, t_f] \;=\; -m(t_f) \quad \rightarrow \quad \text{min!} \tag{43}$$

Here, the direct collocation method was applied on a rather bad initial estimate of the optimal trajectory. For the states, the boundary values have been interpolated linearly and the controls have been set to zero. The direct collocation method DIRCOL converges in two macro iteration steps to a solution with 21 grid points. From this solution, the optimal states and the adjoint variables have been estimated. Based on this estimate, the multiple shooting method was applied to solve the boundary value problem arising from the optimality conditions (see [7]). The final solutions are $m(t_f) = 321243.$ kg and $t_f = 179.75$ s. For these values, the solution of the direct collocation method was accurate to four digits.

In Figs. 1 to 6, the solution of the direct collocation method is shown by a dashed line and the highly accurate solution of the multiple shooting method is shown by a solid line. In the figures, there is no visible difference between the suboptimal and the optimal state variables. Also, the estimated adjoint variables and the suboptimal controls of the direct collocation method show a pretty good conformity with the highly accurate ones. The approximation quality can furthermore be improved by increasing the number of grid points to more than 21. The optimal throttle setting δ equals one within the whole time interval as it is found by both methods.

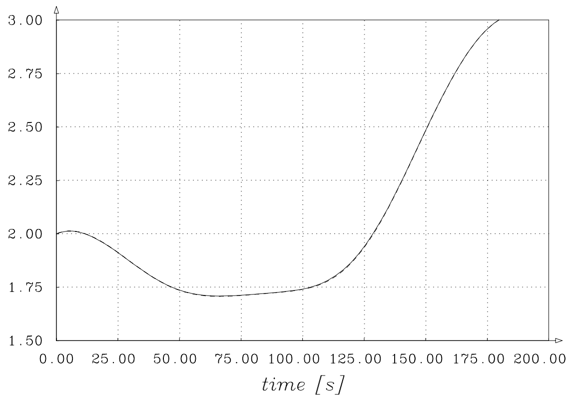

Fig. 1: The altitude h[10km].

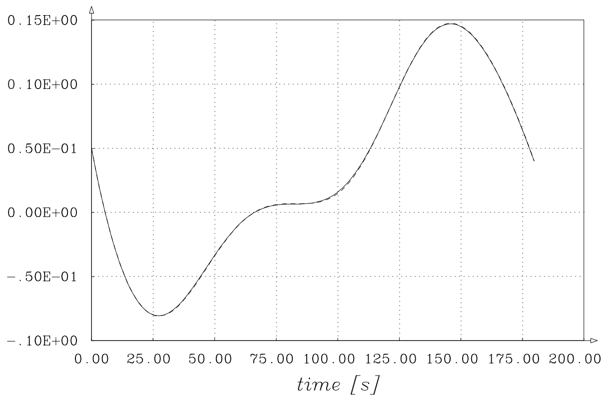

Fig. 2: The flight path angle γ[1].

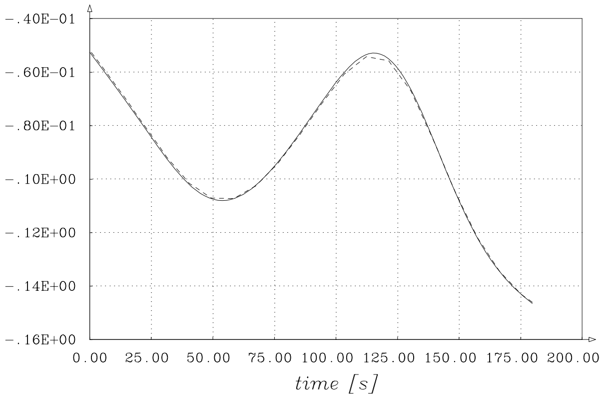

Fig. 3: The adjoint variable λ_h.

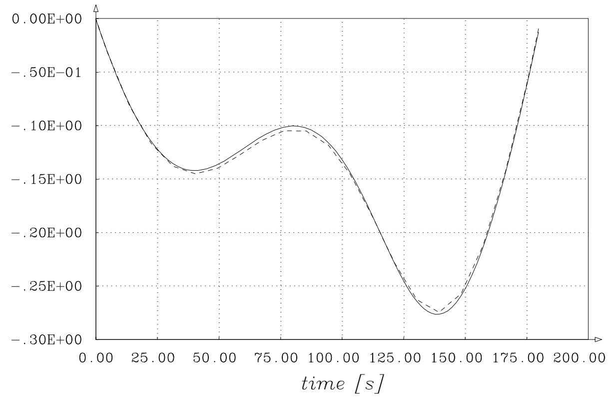

Fig. 4: The adjoint variable λ_γ.

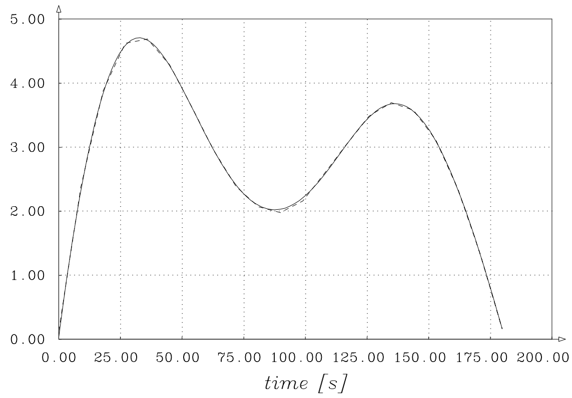

Fig. 5: The lift coefficient c_L.

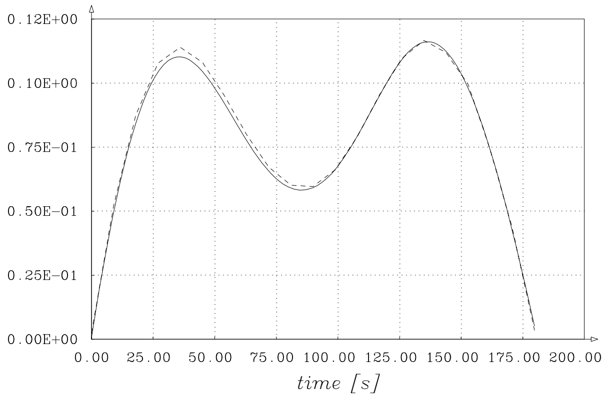

Fig. 6: The thrust angle ϵ[1].

5.2 A problem with a second order state variable inequality constraint

This well-known problem is due to Bryson, Denham, and Dreyfus [3]. After a transformation, the differential equations and boundary conditions are

$$\begin{array}{rclrclrcl} \dot{x} &=& v, & x(0) &=& 0, & x(1) &=& 0, \\ \dot{v} &=& u, & v(0) &=& 1, & v(1) &=& -1, \\ \dot{w} &=& u^2/2, & w(0) &=& 0, & w(1) & \text{is} & \text{free.} \end{array} \tag{44}$$

The objective is

$$J[u] \;=\; w(1) \quad \rightarrow \quad \text{min!} \tag{45}$$

The state constraint to be taken into account is of order 2 here

$$g(x) \;=\; l - x(t) \;\geq\; 0, \quad \frac{\partial}{\partial u}\left(\frac{\mathrm{d}}{\mathrm{d}t} g\right) \;\equiv\; 0, \quad \frac{\partial}{\partial u}\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} g\right) \;\not\equiv\; 0. \tag{46}$$

Explicit formulae of the solution depending on the value of l can be given, cf. [3], [4]. For $l = 1/9$, there exists an interior boundary arc $[t_{\text{entry}}, t_{\text{exit}}] = [t_{\text{I}}, t_{\text{II}}] = [3l, 1-3l]$ where the state constraint is active. The minimum objective value is $w(1) = 4/(9l) = 4$.

With the Hamiltonian $H = \lambda_x v + \lambda_v u + \lambda_w u^2/2 + \eta(l - x)$ the minimum principle yields for the adjoint variables

$$\lambda_x(t) = \begin{cases} 2/(9l^2), & 0 \le t < t_{\mathrm{I}}, \\ 0, & t_{\mathrm{I}} \le t < t_{\mathrm{II}}, \\ -2/(9l^2), & t_{\mathrm{II}} \le t \le 1, \end{cases} \quad \lambda_v(t) = \begin{cases} 2\,(1 - t/(3l))\,/(3l), & 0 \le t < t_{\mathrm{I}}, \\ 0, & t_{\mathrm{I}} \le t < t_{\mathrm{II}}, \\ 2\,(1 - (1-t)/(3l))\,/(3l), & t_{\mathrm{II}} \le t \le 1, \end{cases} \tag{47}$$

and $\lambda_w \equiv 1$. The adjoint variable λ_x suffers discontinuities when entering or leaving the state constraint. A first solution is obtained by using DIRCOL with an equidistant grid of $N = 11$ grid points resulting in a minimum objective value of $w(1) = 3.99338$. In Figs. 7 to 10 these first suboptimal solutions are shown by dashed lines and the exact solutions are shown by solid lines. In addition, the grid points of the discretization are marked.

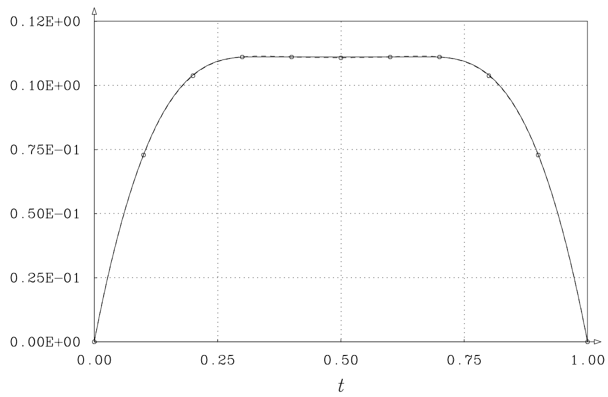

Fig. 7: The state variable x.

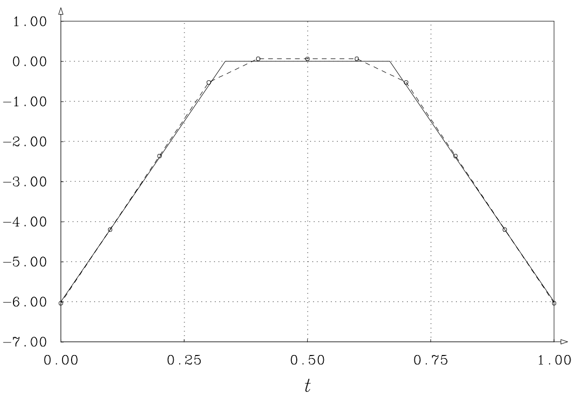

Fig. 8: The control variable u.

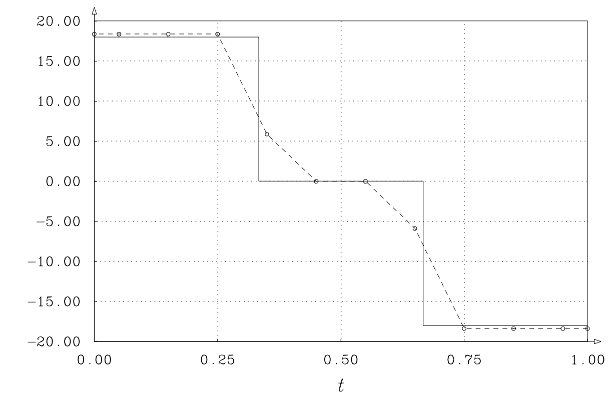

Fig. 9: The adjoint variable λ_x.

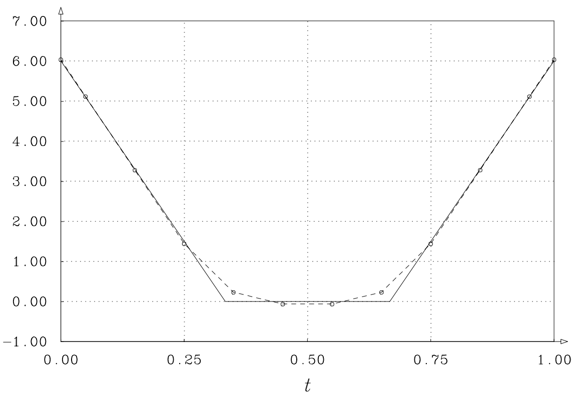

Fig. 10: The adjoint variable λ_v.

The solution is now refined by using a "three-stage" collocation approach that includes the switching structure of the state constraint, i. e. the switching points t_{I} and t_{II} are included as two additional parameters with two additional equality conditions in the optimization procedure

$$g(x) \begin{cases} \geq 0, & 0 \leq t < t_{\mathrm{I}}, \\ = 0, & t_{\mathrm{I}} \leq t < t_{\mathrm{II}}, \\ \geq 0, & t_{\mathrm{II}} \leq t \leq 1, \end{cases} \qquad x(t_{\mathrm{I}} - 0) = l, \qquad x(t_{\mathrm{II}} + 0) = l. \tag{48}$$

The method DIRCOL is now applied to the reformulated problem with a separate grid of 4 grid points in each of the three stages $[0, t_{\mathrm{I}}]$, $[t_{\mathrm{I}}, t_{\mathrm{II}}]$, and $[t_{\mathrm{II}}, 1]$. This results in a minimum objective value of $w(1) = 3.99992$ and a more accurately satisfied state constraint. In Figs. 11 to 14 the refined solutions are shown. In addition, two dotted vertical lines show the entry and exit points of the state constraint that are computed with an error of one percent. The quality of the estimated adjoint variables and also of the control variable has been significantly improved while the dimension of the resulting nonlinear program has not been increased.

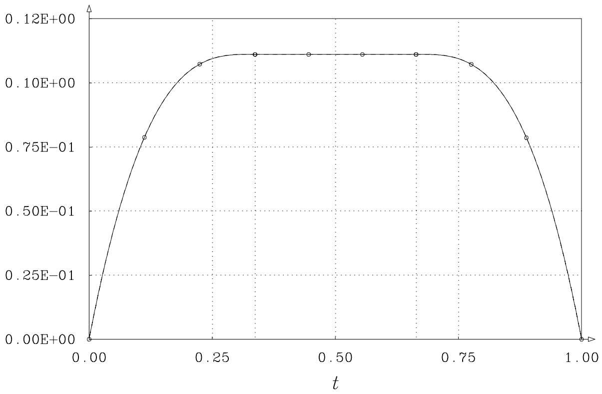

Fig. 11: The state variable x.

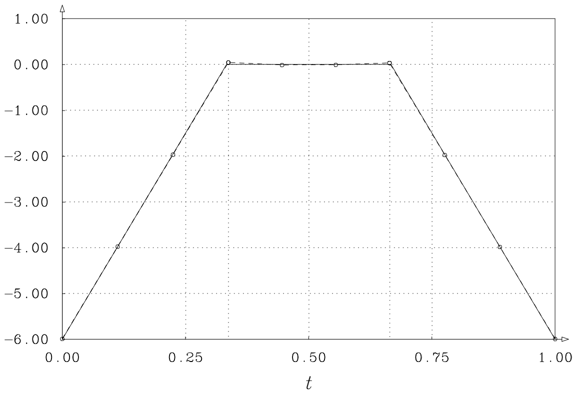

Fig. 12: The control variable u.

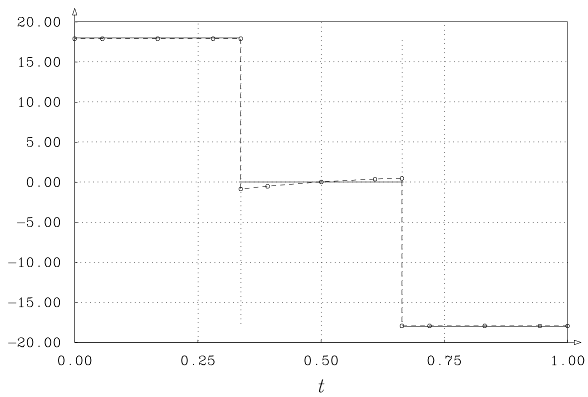

Fig. 13: The adjoint variable λ_x.

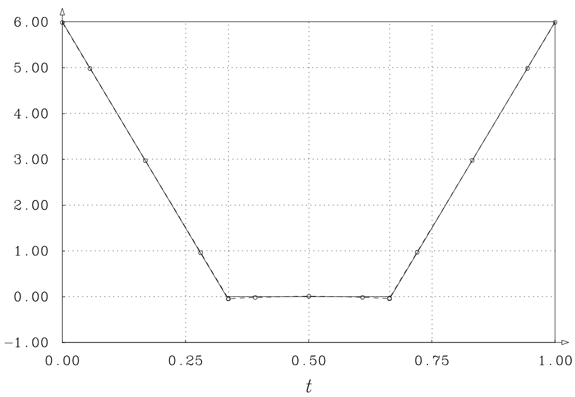

Fig. 14: The adjoint variable λ_v.

6 Conclusions

A way of estimating adjoint variables of optimal control problems by a direct collocation method has been described. The method seems to be superior to previous approaches for estimating adjoint variables in terms of reliability and the ability to include discontinuities in adjoints at the junction points of state constraint subarcs. Furthermore, the estimates of the adjoint variables and the suboptimal controls have been improved by including the switching structure of active state constraints in the optimization procedure.

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